Epsilon-invariant substitutions and indefinite descriptions

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Abstract

It is known that an epsilon-invariant sentence has a first-order reformulation, although it is not in an explicit form, since, the proof uses the non-constructive interpolation theorem. We make an attempt to describe the explicit meaning of sentences containing epsilon-terms, adopting the strong assumption of their first-order reformulability. We will prove that, if a monadic predicate is syntactically independent from an epsilon-term and if the sentence obtained by substituting the variable of the predicate with the epsilon-term is epsilon-invariant, then the sentence has an explicit first-order reformulation. Finally, we point out that the formula gives a contextual-quantificational meaning for the indefinite descriptions, provided that one accepts Kneebone's read of epsilon-terms.

Keywords: Epsilon-calculus, choice logic, epsilon-invariance, substitution, descriptions.

1 Introduction

1.1 The explanation of the problem

G. T. Kneebone, B. Hartly Slater and others pointed out that Hilbert's epsilon-calculus is an appropriate and natural environment for Russell's Theory of Descriptions.[1] In [20], it is showed that if the ε operator is used and the epsilon-axioms are assumed, then Russell's theory becomes in a certain sense complete. Although, in the formal language, Russell's descriptions are not proper names, they are incomplete symbols and have only contextual meaning,[2] Hilbert's epsilon-operator is defined for every formula φ regardless whether or not the extension of φ is a singleton. Hartly Slater[3] showed that the term,

$$t=(\varepsilon x)(\varphi\wedge(\forall y)(\varphi[y/x]\rightarrow x=y)),$$

is a formalized definite description and the following equivalence holds:

$$(\exists x)(\varphi\wedge(\forall y)(\varphi[y/x]\rightarrow x=y)) \quad\vdash\quad \vartheta[t/x]\leftrightarrow(\exists x)(\varphi\wedge(\forall y)(\varphi[y/x]\rightarrow x=y)\wedge\vartheta).$$

The right-hand side of the $\leftrightarrow$ is the well-known Russellian formula, thus it is the contextual meaning of the sentence $\vartheta[t/x]$, provided that the extension of φ is a singleton. We continue this investigation and, adopting the strong assumption of their first-order reformulability, make an attempt to describe

[1] See [10, 19].

[2] According to Russell and Whitehead:

> By an 'incomplete symbol' we mean a symbol which is not supposed to have any meaning in isolation, but is only defined in certain contexts. [22, p. 217].

[3] See [20, p. 417].

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the contextual meaning of sentences of the form $\vartheta[(\varepsilon x)\varphi/x]$, where $(\varepsilon x)\varphi$ is an arbitrary epsilon-term. In Section 3 we will prove that, if Γ consists of epsilon-invariant sentences and the formulas φ and ϑ are independent from the term $(\varepsilon x)\varphi$ (in a sense[4]), then $\Gamma \vdash \vartheta[(\varepsilon x)\varphi/x]$ will be equivalent to

$$\Gamma \vdash ((\exists x)\varphi \wedge (\forall x)(\varphi \rightarrow \vartheta)) \vee (\neg(\exists x)\varphi \wedge (\forall x)\vartheta)$$

in the so-called intensional epsilon-calculus. Obviously, the result is a theorem of the epsilon-logic, despite that the motivation and some of the conclusions belong to the philosophy of language or the philosophy of logic.

1.2 The problem in a wider view

Bertrand Russell, in his Theory of Descriptions, dealt with the meaning of natural language sentences of the form:

$$\text{'The } F \text{ is } G\text{.'} \qquad \text{(DD)}$$

He argued that the phrase 'the F', which is called a definite description, is an incomplete expression in the sense that it has meaning only in the context of (DD).[5] According to Russell, the meaning of (DD) is

$$\text{'There is one and only one } F\text{, and that one is } G\text{.'} \qquad \text{(RU1)}$$

The above proposal seems to solve two famous problems in the philosophy of language. The first one is the meaning of non-referring descriptions whereas the second is Frege's modal logical puzzle on Hesperus and Phosphorus.[6]

A noteworthy property of Russell's proposal is that it is a solution within the first-order logic. Indeed, if the predicates F and G in the meta-language correspond to the formulas φ and ϑ, respectively, then the Russellian proposal associates the sentence (DD) with the following formula:

$$(\exists x)(\vartheta \wedge (\forall y)(\varphi[y/x] \leftrightarrow x = y)). \qquad \text{(RU2)}$$

Russell's description elimination program, i.e. to treat definite descriptions according to (RU2), became a doctrine in formal logic, and the Theory of Descriptions and the formalization of the natural language of mathematics are closely intertwined from the beginning.

The claim that (DD) means (RU1) is a thesis in the philosophy of language, but using the reformulation (RU2), one could make an attempt to prove the thesis in a strict and formal way in formal languages containing descriptive terms. For example, if the language of the first-order modal logic is extended by a descriptor operator ι and the descriptive terms $(\iota x)\varphi$, then the proposition,

$$\vartheta((\iota x)\varphi) \leftrightarrow (\exists x)(\vartheta \wedge (\forall y)(\varphi[y/x] \leftrightarrow x = y)),$$

[4]See Definition 3.3.
[5]See [22].
[6]See [17]. The original problems can be found in [7].

is valid according to the modal semantics, at least in a modal form. Both in [6] and in [18], it is discussed, however in slightly different ways. These investigations prefer Frege's view and consider definite descriptions as complete symbols.

Also, there are formalizational projects which deviate from the Russellian doctrine and instead prefer Hilbert's pre-suppositional solution of the problem of definite descriptions. Such formal languages can be found in Hilbert and Bernays' *Grundlagen der Mathematik* and in Bourbaki's *Éléments de mathématique*.[7] In the epsilon-languages, extra expressions of the form:

$$(\varepsilon x)\varphi$$

called the *epsilon-terms* are added. According to Slater[8], if

$$\varphi \wedge (\forall y)(\varphi[x/y] \to x = y)$$

is denoted by ψ, then it can be easily proved that

$$\{(\exists x)\psi\} \quad \vdash \quad \vartheta[(\varepsilon x)\psi/x] \leftrightarrow (\exists x)(\varphi \wedge (\forall y)(\varphi[y/x] \to x = y) \wedge \vartheta), \tag{PS}$$

where $(\exists x)\psi$ is the pre-supposition for the Russellian definite description to be expressible in the formal language. Obviously, if the pre-supposition above holds, then the formal expression $(\varepsilon x)\psi$ corresponds to both the Russellian and the Hilbertian definite description. Note that, if the pre-supposition does not hold, then $(\varepsilon x)\psi$ is neither a Russellian, nor a Hilbertian definite description; it is rather a non-attributive description in the sense of Donnellan's definition.[9]

The present article's main problem arises here. If we turn to the case of general epsilon-terms, the question is the following. Is there a general counterpart of (PS) for an arbitrary epsilon-term, for a ψ which does express neither unicity nor existence, provided that $\vartheta[(\varepsilon x)\psi/x]$ has a first-order reformulation.

1.3 The proposed solution

For us, it is important that the operations used in (RU2) (i.e. quantification, identity, Boolean operations, etc.) are solely logical ones, hence (RU2) could be considered a *purely logical* sentence. In the Tarskian sense, a notion is *logical* if it is a composition of permutation-invariant operations such as quantification, identity, substitution of variables and so on.[10] As opposed to Russell's descriptor, Hilbert's operator does not satisfy the permutation-invariant property. The sentence,

$$\vartheta((\varepsilon x)\varphi),$$

is not necessarily purely logical, since generally, it does not have a plain first-order reformulation. Our goal is to find the first-order reformulation of the sentence $\vartheta((\varepsilon x)\varphi)$ and its meaning, provided the sentence $\vartheta((\varepsilon x)\varphi)$ is purely logical.

Our hypothesis is that, if there is any purely logical meaning of $\vartheta[(\varepsilon x)\varphi/x]$, then it must be a weak form of the purely logical meaning of $\vartheta[(\iota x)\varphi/x]$ proposed by Russell. Since, we follow Tarski, on

[7] See [4, 8].
[8] See [20].
[9] See [5].
[10] See [21].

'purely logical' we mean plain first-order reformulation, actually it will be enough to work with a certain type of epsilon-invariance. As it is known, if $\vartheta[(\varepsilon x)\varphi/x]$ is epsilon-invariant then, according to Blass and Gurevich,[11] it has a first-order reformulation. But, is there an explicit reformulation of $\vartheta[(\varepsilon x)\varphi/x]$? To answer the question, let us consider the sentence 'The F is G.' and recall Neale's version of Russell's proposal[12]

'There is an F, there is at most one F, and *every* F is G.'

Since, we think of epsilon-terms as not necessarily definite descriptions, first, let us erase the uniqueness clause:

'There is an F, and every F is G.'

The schema obtained just now is obviously not applicable to the epsilon-operator, since there are sentences containing epsilon-terms which are true, whereas the existence formula associated to the epsilon-term is false. For example,

$$(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)$$

is true, however,

$$(\exists x)(x \neq x)$$

is false. Similar effects occur when the property in question is true for every individuals, thus it seems sound to express our proposal in the following form:

'There is an F and every F is G, or there is no F, however everything is G.' (M)

In Theorem 3.8 (Section 3) we will conclude that (M) can be considered to be the meta-language translation of the formal sentence $\vartheta[(\varepsilon x)\varphi/x]$, if ϑ and φ both are independent from $(\varepsilon x)\varphi$ in the sense of Definition 3.3 (Section 3).

We emphasize that, while the Theory of Descriptions is a part of linguistics and philosophy, exploring the model-theoretical meaning of sentences in the epsilon-language is a task of mathematical logic.

2 Preliminaries

2.1 Definitions: syntax

In the literature there are two versions of epsilon-languages, for later purposes it is convenient to consider both of them. Let $\mathsf{t} = (r_i, f_j, c_k)_{ijk}$ be a similarity type and let $\mathcal{L}_\exists$ be the usual first-order language generated freely with respect to the operators $\neg, \vee, \exists, \mathsf{t}$. $\mathcal{L}_\exists$ can be extended to the language,

$$\mathcal{L}_{\exists,\varepsilon},$$

[11] See Theorem 2.29.
[12] See [15, p. 21].

by adding the terms $(\varepsilon x)\varphi$ to the class Term($\mathcal{L}_\exists$), where x is any variable, and φ is any formula. If φ is a one-variable formula of the variable x, then $(\varepsilon x)\varphi$ is called the epsilon-term *associated to* φ. Sometimes, $\mathcal{L}_{\exists,\varepsilon}$ is called *the language of the predicate calculus with epsilon* and it is accurately defined, e.g., in [13][13] or in [12].[14]

Another epsilon-language is obtained when one extends the language of *the elementary calculus*[15] by adding the epsilon-terms. If $\mathcal{L}$ denotes the first-order language of type t without quantifiers, then let

$$\mathcal{L}_\varepsilon$$

be the extension of $\mathcal{L}$ by the epsilon-terms. It is clear that $\mathcal{L}_\varepsilon$ is the freely generated language with respect to the operators $\neg, \vee, \varepsilon, \mathsf{t}$ and it is called *the language of the elementary calculus with epsilon*.[16]

2.2 Definitions and facts: calculus

A featured connection arises between $\mathcal{L}_{\exists,\varepsilon}$ and $\mathcal{L}_\varepsilon$ when we consider the axioms of the epsilon-calculus.

Definition 2.1 (PC$_\varepsilon$)
Let PC$_\varepsilon$ be the deductive system of predicate logic with equality in the language $\mathcal{L}_{\exists,\varepsilon}$ with the following *critical axioms*:

$$\varphi[t/x] \to \varphi[(\varepsilon x)\varphi/x], \tag{CA}$$

where $[./x]$ is the operation of substitution, x is a variable, $t \in \text{Term}(\mathcal{L}_{\exists,\varepsilon})$ is any term and $\varphi \in \text{Form}(\mathcal{L}_{\exists,\varepsilon})$ is a formula (t is free for x in φ).

Remark 2.2
The critical axioms imply the ones below:

$$\vdash_{\text{PC}_\varepsilon} (\exists x)\varphi \leftrightarrow \varphi[(\varepsilon x)\varphi/x], \qquad \vdash_{\text{PC}_\varepsilon} (\forall x)\varphi \leftrightarrow \varphi[(\varepsilon x)\neg\varphi/x].$$

Definition 2.3 (EC$_\varepsilon$)
The elementary calculus with epsilon in the language $\mathcal{L}_\varepsilon$ (thus without explicit quantifiers), is the deductive system of the predicate logic (EC)—including the equality rules and without the quantifiers—extended by the schema (CA) and the following (Gen) as axioms:

$$\varphi \to \varphi[(\varepsilon x)\neg\varphi/x], \tag{Gen}$$

where x does not occur in φ. It is denoted by EC$_\varepsilon$.

Remark 2.4
Since $(\forall x)\varphi$ is defined to be $\varphi[(\varepsilon x)\neg\varphi/x]$ in EC$_\varepsilon$, after a time, (Gen) becomes the usual $\varphi \to (\forall x)\varphi$ schema.

[13] The Epsilon-Calculus: Syntax, Sec. 2, [14].
[14] [12, The Hilbert ε-operator, p. 481], aside from the operator **O**.
[15] The elementary calculus is the first-order logical calculus without quantifiers and the rules of quantification. It is denoted by EC.
[16] A more formal definition of $\mathcal{L}_\varepsilon$ can be found in [14, Sec. 2].

Now, let

$$(.)^{*\varepsilon} : \mathcal{L}_{\exists,\varepsilon} \to \mathcal{L}_{\varepsilon}$$

be the language homomorphism such that $(.)^{*\varepsilon}$ preserves $\neg, \vee, \varepsilon, \mathfrak{t}$ and the following correspondence:

$$((\exists x)\varphi)^{*\varepsilon} = \varphi^{*\varepsilon}[(\varepsilon x)\varphi^{*\varepsilon}/x].$$

Proposition 2.5 (Moser–Zach)
$(.)^{*\varepsilon}$ is a $\mathrm{PC}_{\varepsilon} \to \mathrm{EC}_{\varepsilon}$ embedding which preserves the provability.[17]

Remark 2.6
Hence, it is clear that there is a canonical monomorphism

$$(.)^{\varepsilon} : \mathcal{L}_{\exists} \to \mathcal{L}_{\varepsilon},$$

which is invariant with respect to the operators $\neg, \vee, \mathfrak{t}$ and the correspondence $((\exists x)\varphi)^{\varepsilon} = \varphi^{\varepsilon}[(\varepsilon x)\varphi^{\varepsilon}/x]$. This leads to the fact that PC can also be embedded into $\mathrm{EC}_{\varepsilon}$.

Remark 2.7
Note that Moser and Zach's result above is an exact form of the fact that was proved by Bourbaki in an informal way. (In their work, the French group of mathematicians built the predicate calculus onto the epsilon-language without any explicit quantifiers.) Nevertheless, the embedding result has long been a part of the mathematicians' folklore.

2.3 *Definitions and facts: intensional semantics*

The first model-theoretic semantics goes back to Günter Asser. A complete description of the semantics can be found in [1, 2]. The latter article contains a remarkable completeness proof in the context of automated theorem proving. In the following, we describe the so-called intensional and the well-known extensional semantics briefly.

Remark 2.8
The authors of Gundlagen did not give any formal semantics to explain their calculus. It is not really surprising since, at that time, the model-theoretical semantics were not developed well enough. Just for that, Hilbert's terms were the objects which filled the lack of semantic reference and played the role of the members of the universe of a future canonical model. Indeed, if one compares the epsilon-terms to the Henkin witnesses,[18] one can find that the Henkin witness c of the valid sentence $(\exists x)\varphi(x)$ corresponds to the epsilon-term associated to $\varphi(x)$, since they both satisfy the predicate $\varphi(x)$. Hence, the formulas

$$(\exists x)\varphi(x) \to \varphi(c) \quad \text{and} \quad (\exists x)\varphi(x) \to \varphi((\varepsilon x)\varphi(x))$$

hold. Using the above property of the epsilon-terms, Ackermann, Hilbert and Bernays were able to prove the consistency of some certain formal theories.

[17] For more details, see [14, The Embedding Lemma, Sec. 4].

[18] Henkin witnesses are mentioned in the proof of the Compactness Theorem or the Omitting Types Theorem of FOL. If a formula $(\exists x)\varphi(x)$ holds and for a term c, then formula $\varphi(c)$ holds too, and c is a Henkin witness for the existential formula $(\exists x)\varphi(x)$. See [9, pp. 265, 334].

Epsilon-terms are closely related to Skolem functions,[19] hence we will define the $\mathcal{L}_\varepsilon$-structures by special Skolem expansions of the usual first-order models. By Skolem expansions, we mean models of a first-order language extended by Skolem symbols, where the interpretations of Skolem symbols are Skolem functions (the definitions can be found in [12, 11.33–36, pp. 211–2]).

FACT 2.9
Let us add the Skolem function symbol $S_{(\exists x)\varphi}$ to the language $\mathcal{L}_\exists$ for every existential formula $(\exists x)\varphi$. If $\mathcal{L}'_\exists$ denotes the Skolem extension of $\mathcal{L}_\exists$, then there is a *canonical language monomorphism*

$$(.)^* : \mathcal{L}_{\exists,\varepsilon} \to \mathcal{L}'_\exists,$$

which sends the epsilon-terms to the Skolem expressions, i.e.

$$((\varepsilon x)\varphi)^* = S_{(\exists x)\varphi} v_1 \dots v_k,$$

where $k = \max\{i \mid v_i \in \mathrm{FreeVar}((\exists x)\varphi)\}$. (Cf. [12, Def. 29.23, p. 481].)

REMARK 2.10
The first problem of constructing a semantics for the epsilon-terms turns up at this point. It is clear that, in the Skolem expansion $\mathfrak{N}$

$$\big(\big(S_{(\exists x)\varphi} v_1 \dots v_k\big)[t/v_i]\big)^{\mathfrak{N}}[a] = \big(S_{(\exists x)\varphi} v_1 \dots v_k\big)^{\mathfrak{N}}\big[a_i^n\big],$$

where term t is free for v_i in $(\exists x)\varphi$, $a \in {}^\omega N$, $n = t^{\mathfrak{N}}[a]$, and the sequence $(a \setminus \{(i, a_i)\}) \cup \{(i, n)\}$ is denoted by a_i^n. But, for us, the Skolem terms (as the (.)*-images of the epsilon-terms) are needed to show the following property:

$$\big(S_{(\exists x)\varphi[t/v_i]} v_1 \dots v_k\big)^{\mathfrak{N}}[a] = \big(S_{(\exists x)\varphi} v_1 \dots v_k\big)^{\mathfrak{N}}\big[a_i^n\big]. \tag{Sub}$$

Property (Sub) is the same as substitutivity in [1, Def. 5] and this one is so important that we redefine it in the context of the method of Skolem functions.

DEFINITION 2.11
Let $\mathfrak{M}$ be a first-order model. The Skolem expansion $\mathfrak{N}$ of $\mathfrak{M}$ is *substitutive* (*substitutive epsilon-stucture*), if for every $(\exists x)\varphi$ and $a \in {}^\omega M$

$$\big(S_{(\exists x)\varphi[t/v_i]} v_1 \dots v_k\big)^{\mathfrak{N}}[a] = \big(S_{(\exists x)\varphi} v_1 \dots v_k\big)^{\mathfrak{N}}\Big[a_i^{t^{\mathfrak{N}}[a]}\Big]$$

for every term t such that t is free for v_i in $(\exists x)\varphi$. (Cf. [1, Def. 5].)

REMARK 2.12
Note that, Ahrendt and Giese introduced several types of epsilon-structures. The structures expanded by Skolem functions and the substitutive Skolem expansions correspond to the so-called intensional and substitutive structures, respectively (see [1, Def. 4,5]).

The solution of the problem of substitutivity is to introduce epsilon-matrices.

[19] It is pointed out in [12, The Hilbert ε-operator, p. 481] and in [11, Sec. 2.: Quantifier-Free Extensions of Formulas and ε-Theorems)].

DEFINITION 2.13 (Epsilon-matrix)
Let us suppose that the epsilon-term t has an occurrence in the formula φ. If this occurrence of t is also an occurrence of t in another epsilon-term s occurring in φ, then it is said to be *interior* in φ, otherwise it is an *exterior* occurrence of t in φ. The epsilon-term

$$(\varepsilon x)\psi(x, y_1, \ldots, y_n)$$

is *a matrix of the epsilon-term* $(\varepsilon x)\varphi$ if

(1) $y_1, \ldots, y_n \notin \mathrm{Var}((\varepsilon x)\varphi)$;
(2) $\psi(x, y_1, \ldots, y_n)$ is an $(n+1)$-variable formula; and
(3) the distinct epsilon-terms $t_1, \ldots, t_k$ have exterior occurrence in φ such that

$$(\varepsilon x)\varphi = (\varepsilon x)\psi(x, y_1, \ldots, y_n)[t_1/y_1, \ldots, t_n/y_n].$$

(Cf. [11, p. 138]).

REMARK 2.14
It is known that for $(\varepsilon x)\varphi$ there are unique $\psi(x, y_1, \ldots, y_n), t_1, \ldots, t_n$ with the above three properties. Here, 'unique' means that up to the change of variables $y_1, \ldots, y_n$ and the simultaneous change of x in $(\varepsilon x)\varphi$ and $(\varepsilon x)\psi(x, y_1, \ldots, y_n)$ (Cf. [13, Sec. 3, p. 21]). The relation '$(\varepsilon x)\varphi$ and $(\varepsilon y)\vartheta$ have the same epsilon-matrix' is an equivalence relation between the epsilon-terms.

DEFINITION 2.15
The equivalence class containing $(\varepsilon x)\varphi$ is denoted by

$$\mathrm{mat}((\varepsilon x)\varphi)$$

and the set of all equivalence classes is denoted by

$$\mathrm{Mat}.$$

REMARK 2.16
Note that $\mathrm{mat}((\varepsilon x)\varphi)$ is substitution-invariant in the sense that, if $(\varepsilon x)\psi$ is a matrix of $(\varepsilon x)\varphi$ and t is an epsilon-term such that $x \notin \mathrm{BoundVar}(t)$, then

$$(\varepsilon x)\psi[t/v] \in \mathrm{mat}((\varepsilon x)\varphi).$$

The definition below is due to Moser and Zach, and it is presented at the 17th Computer Science Logic Workshop and 8th Kurt Gödel Colloquium (Vienna, 2003).

DEFINITION 2.17 (Model)
Let t be a similarity type and $\mathfrak{M}$ be a first-order model of type t. Let $\mathcal{P}(M)$ be the set of all subsets of M and let ${}^{[\omega]}M$ be the set of finite sequences in M. *A model of the language* $\mathcal{L}_\varepsilon$ (or an *$\mathcal{L}_\varepsilon$-structure*) is a pair $(\mathfrak{M}, f)$ where the function f satisfies the property

$$f : \mathrm{Mat} \times \mathcal{P}(M) \times {}^{[\omega]}M \longrightarrow M,$$
$$f(\mathrm{mat}((\varepsilon v_i)\varphi), S, (a_1, \ldots, a_n)) \in S, \quad \text{if } S \in \mathcal{P}(M) \setminus \{\varnothing\}.$$

We will denote the class of all such models of an epsilon-language $\mathcal{L}_\varepsilon$ by $\mathsf{Int}(\mathcal{L}_\varepsilon)$.

Remark 2.18

In order to define the satisfaction in $(\mathfrak{M},f)$, we will consider Monk's observation and we will use the correspondence $(.)^*:\mathfrak{L}_{\exists,\varepsilon}\to\mathfrak{L}'_\exists$ between the language $\mathfrak{L}_{\exists,\varepsilon}$ and the Skolem expansion $\mathfrak{L}'_\exists$ of the language $\mathfrak{L}_\exists$ (Cf. [12, Prop. 29.24, p. 482]). Note that $\mathfrak{L}_\varepsilon\subseteq\mathfrak{L}_{\exists,\varepsilon}$ and $(.)^*$ sends the epsilon-terms of $\mathfrak{L}_\varepsilon$ also to Skolem terms.

Definition 2.19 (Satisfaction)

Let $(\mathfrak{M},f)$ be a model of the epsilon-language $\mathfrak{L}_\varepsilon$. First we define the Skolem expansion $\mathfrak{M}_f$ of $\mathfrak{L}_\exists$. $\mathfrak{M}$ is a reduct of $\mathfrak{M}_f$ i.e. $\mathfrak{M}_f\restriction\mathfrak{L}_\exists=\mathfrak{M}$ and the interpretations of the Skolem terms are as follows. If $a\in{}^\omega M$, then

$$\left(\mathrm{S}_{(\exists v_i)\varphi}v_1\ldots v_k\right)^{\mathfrak{M}_f}[a]=f\left(\mathrm{mat}((\varepsilon v_i)\varphi),\left\{m\in M\mid\mathfrak{M}_f\models\varphi\left[a_i^m\right]\right\},\left(t_1^{\mathfrak{M}_f}[a],\ldots,t_n^{\mathfrak{M}_f}[a]\right)\right),$$

where $t_1,\ldots,t_n$ are obtained by representing the epsilon-term $(\varepsilon v_i)\varphi$ in its matrix form: $(\varepsilon v_i)\varphi=(\varepsilon v_i)\psi(v_i,t_1,\ldots,t_n)$.

Finally, the meanings of the terms t and formulas φ of $\mathfrak{L}_\varepsilon$ under the valuation a in $(\mathfrak{M},f)$ are defined to be

$$t^{(\mathfrak{M},f)}[a]=t^{*\mathfrak{M}_f}[a]\quad\text{and}\quad(\mathfrak{M},f)\models\varphi[a]\text{ iff }\mathfrak{M}_f\models\varphi^*[a].$$

An immediate consequence of the usual Substitution Lemma is that every model $(\mathfrak{M},f)$ can be constructed from a substitutive Skolem expansion and vice versa. We state and prove the Substitution Lemma for the epsilon-logic.

Proposition 2.20 (Substitution Lemma)

Let $(\mathfrak{M},f)$ be a model of the epsilon-language $\mathfrak{L}_\varepsilon$, $\varphi\in\mathrm{Form}(\mathfrak{L}_\varepsilon)$, $t,s\in\mathrm{Term}(\mathfrak{L}_\varepsilon)$, $k\in\omega$, $a\in{}^\omega M$, $u=t^{(\mathfrak{M},f)}[a]$ and t is free for v_k in φ and s. Then

$$(\mathfrak{M},f)\models\varphi[t/v_k][a]\quad\text{iff}\quad(\mathfrak{M},f)\models\varphi[a_k^u],\qquad(s[t/v_k])^{(\mathfrak{M},f)}[a]=s^{(\mathfrak{M},f)}[a_k^u].$$

Proof. By structural induction. The only non-trivial case is the induction step with the epsilon-terms. Suppose that t is an epsilon-term and $s=(\varepsilon v_i)\varphi=(\varepsilon v_i)\psi(v_i,t_1,\ldots,t_n)$ with its matrix $(\varepsilon v_i)\psi(v_i,y_1,\ldots,y_n)$. Since v_i is bound in s, let us assume without the loss of generality that $v_i\neq v_k$. Thus, it follows that $s[t/v_k]=(\varepsilon v_i)(\varphi[t/v_k])$. Note that if t occurs in $\psi(v_i,t_1,\ldots,t_n)[t/v_k]$, then it is an interior occurrence, therefore $(\varepsilon v_i)\psi(v_i,t_1,\ldots,t_n)[t/v_k]\in\mathrm{mat}((\varepsilon v_i)\varphi)$. Furthermore, by the induction hypothesis

$$t_1[t/v_k]^{(\mathfrak{M},f)}[a]=t_1^{(\mathfrak{M},f)}[a_k^u],\ \ldots,\ t_n[t/v_k]^{(\mathfrak{M},f)}[a]=t_n^{(\mathfrak{M},f)}[a_k^u].$$

For short, let $\mu=\mathrm{mat}((\varepsilon v_i)\varphi)$ and $\mu'=\mathrm{mat}((\varepsilon v_i)\psi(v_i,t_1,\ldots,t_n)[t/v_k])$. As we know $\mu=\mu'$, thus

$$\begin{aligned}((\varepsilon v_i)\varphi[t/v_k])^{(\mathfrak{M},f)}[a]&=f\left(\mu',\{m\in M\mid(\mathfrak{M},f)\models\varphi[t/v_k][a_i^m]\},(t_1[t/v_k]^{(\mathfrak{M},f)}[a],\ldots,t_n[t/v_k]^{(\mathfrak{M},f)}[a])\right)\\&=f\left(\mu',\{m\in M\mid(\mathfrak{M},f)\models\varphi[a_{i\ k}^{m\ u}]\},(t_1^{(\mathfrak{M},f)}[a_k^u],\ldots,t_n^{(\mathfrak{M},f)}[a_k^u])\right)\\&=f\left(\mu,\{m\in M\mid(\mathfrak{M},f)\models\varphi[a_{k\ i}^{u\ m}]\},(t_1^{(\mathfrak{M},f)}[a_k^u],\ldots,t_n^{(\mathfrak{M},f)}[a_k^u])\right)\\&=s^{(\mathfrak{M},f)}[a_k^u].\end{aligned}$$

∎

Here, we do not prove the soundness and completeness property in a straightforward way. Instead, we recall Ahrendt and Giese's construction and we show that the semantics above is the substitutive semantics described in [1].

CONSEQUENCE 2.21
The substitutive Skolem expansions of the first-order model $\mathfrak{M}$ defined in Definition 2.11 are the Skolem expansions $\mathfrak{M}_f$ defined in Definition 2.19.

PROOF. The Substitution Lemma states that every $\mathfrak{M}_f$ is substitutive. Let us consider a substitutive Skolem expansion $\mathfrak{N}$ of $\mathfrak{M}$. We construct a function f such that $\mathfrak{N}=\mathfrak{M}_f$. We show that, for an epsilon-term $(\varepsilon x)\varphi$, $(S_{(\exists x)\varphi}v_1\ldots v_k)^{\mathfrak{N}}$ is determined by the Skolem function of the matrix of $(\varepsilon x)\varphi$. Let $(\varepsilon x)\psi(x,t_1,\ldots,t_k)$ be the matrix representation of $(\varepsilon x)\varphi$. By substitutivity

$$\begin{aligned}(S_{(\exists x)\varphi}v_1\ldots v_n)^{\mathfrak{N}}[a]&=(S_{(\exists x)\psi(x,t_1,\ldots,t_k)}v_1\ldots v_n)^{\mathfrak{N}}[a]\\&=(S_{(\exists x)\psi(x,v_{i_1},\ldots,v_{i_k})}v_1\ldots v_n)^{\mathfrak{N}}\left[a^{t_1^{\mathfrak{N}}[a]}_{i_1}\ldots{}^{t_k^{\mathfrak{N}}[a]}_{i_k}\right].\end{aligned}$$

Now, write v_{i_0} instead of x. Let $f:\mathrm{Mat}\times\mathcal{P}(M)\times{}^{[\omega]}M\longrightarrow M$ be the following:

$$f(\mathrm{mat}((\varepsilon v_{i_0})\psi(v_{i_0},v_{i_1},\ldots,v_{i_k})),S,b)=(S_{(\exists v_{i_0})\psi(v_{i_0},v_{i_1},\ldots,v_{i_k})}v_1\ldots v_n)^{\mathfrak{N}}\left[a^{t_1^{\mathfrak{N}}[a]}_{i_1}\ldots{}^{t_k^{\mathfrak{N}}[a]}_{i_k}\right]$$

for every matrix $(\varepsilon v_{i_0})\psi(v_{i_0},v_{i_1},\ldots,v_{i_k})$ and for every (S,b) of the form:

$$\left(\left\{m\in M\mid\mathfrak{N}\models\psi(v_{i_0},v_{i_1},\ldots,v_{i_k})\left[a^{m}_{i_0}{}^{t_1^{\mathfrak{N}}[a]}_{i_1}\ldots{}^{t_k^{\mathfrak{N}}[a]}_{i_k}\right]\right\},(t_1^{\mathfrak{N}}[a],\ldots,t_k^{\mathfrak{N}}[a])\right),$$

where $t_1,\ldots,t_k$ are arbitrary terms and a is arbitrary valuation, and a simple choice function otherwise. Finally, by induction it can be shown that f is well-defined and $\mathfrak{M}_f=\mathfrak{N}$. ∎

REMARK 2.22
The soundness and completeness hold, i.e. for every set of sentences $\Gamma\cup\{\varphi\}\subseteq\mathrm{Sent}(\mathcal{L}_\varepsilon)$

$$\Gamma\underset{\mathrm{EC}_\varepsilon}{\vdash}\varphi\quad\text{iff}\quad\Gamma\underset{\mathrm{Int}}{\models}\varphi.$$

It is shown in [1, Thm. 4, 5] or it can be shown by the Henkin construction of canonic term model.

The *validity* concerning classes can be defined as follows.

DEFINITION 2.23
Let φ be a sentence in the language $\mathcal{L}_\varepsilon$ of similarity type t. If K is a class of first-order models of type t, then

$$\mathsf{K}\underset{\mathrm{Int}}{\models}\varphi\quad\overset{\mathrm{def}}{\Longleftrightarrow}\quad(\mathfrak{M},f)\models\varphi\text{ for every }(\mathfrak{M},f)\in\mathsf{Int}(\mathcal{L}_\varepsilon)\text{ such that }\mathfrak{M}\in\mathsf{K}.$$

2.4 Definitions and facts: extensional semantics

The semantics above is the so-called *intensional semantics*, however, it is not an intensional system in the sense of modal logic, and it is not only intensional but also substitutive in the sense of [1, Def. 5]. The term was introduced in contrast to the following class of epsilon-models.

Definition 2.24 (Extensional models)
Let $\mathfrak{L}_\varepsilon$ be the language of the elementary calculus with epsilon.

$$\mathsf{Ext}(\mathfrak{L}_\varepsilon)=\{(\mathfrak{M},f)\in\mathsf{Int}(\mathfrak{L}_\varepsilon)\,|\,(\forall t_1,t_2\in\mathrm{Mat})(\forall s_1,s_2\in{}^{[\omega]}M)f(t_1,.,s_1)=f(t_2,.,s_2)\}.$$

(Cf. [12, Def. 29.23, p. 481] and [1, Def. 6].) The members of $\mathsf{Ext}(\mathfrak{L}_\varepsilon)$ are called the *extensional models* or the *dependent choice structures*.

Remark 2.25
The reason for the name is that the reference of an epsilon-term $(\varepsilon x)\varphi$ depends only on the extension of φ.

Definition 2.26 (EEC$_\varepsilon$)
On the *extensional epsilon-calculus*, denoted by EEC$_\varepsilon$, we mean EC$_\varepsilon$ with the following axiom schema:

$$(\forall x)(\varphi\leftrightarrow\psi)\rightarrow(\varepsilon x)\varphi=(\varepsilon x)\psi.$$

Remark 2.27
The extensional semantics is sound and complete with respect to the above calculus,[20] i.e.

$$\Gamma\underset{\mathrm{EEC}_\varepsilon}{\vdash}\varphi\quad\text{iff}\quad\Gamma\underset{\mathrm{Ext}}{\models}\varphi$$

for every $\Gamma\cup\{\varphi\}\subseteq\mathrm{Sent}(\mathfrak{L}_\varepsilon)$. Here $\Gamma\underset{\mathrm{Ext}}{\models}\varphi$ denotes that the sentence φ is valid in every extensional model $(\mathfrak{M},f)$ which is a model of the set of sentences Γ.

Definition 2.28 (Epsilon-invariant formula)
The formula $\varphi\in\mathrm{Form}(\mathfrak{L}_\varepsilon)$ is *epsilon-invariant* if for all $(\mathfrak{M},f),(\mathfrak{M},g)\in\mathsf{Ext}(\mathfrak{L}_\varepsilon)$ and for all $a\in{}^{\omega}M$

$$(\mathfrak{M},f)\models\varphi[a]\quad\text{iff}\quad(\mathfrak{M},g)\models\varphi[a].$$

Blass and Gurevich proved the following theorem concerning epsilon-invariance.

Theorem 2.29 (Blass–Gurevich)
If a formula $\varphi\in\mathrm{Form}(\mathfrak{L}_\varepsilon)$ is epsilon-invariant, then there is a first-order (epsilon-free) formula $\psi\in\mathrm{Form}(\mathfrak{L}_\exists)$ such that

$$\varphi^{(\mathfrak{M},f)}=\psi^{\mathfrak{M}}$$

holds for all $(\mathfrak{M},f)\in\mathsf{Ext}(\mathfrak{L}_\varepsilon)$. (Cf. [3, Prop. 3.2])

Remark 2.30
In a sense, the theorem states that the epsilon-invariant formulas are the formulas in which the ε operators are used solely for quantification. However, generally there is hardly any chance to find the first-order formula ψ above, since the proof uses Craig's non-constructive interpolation theorem.

Epsilon-invariance might be introduced in a more specific way.

[20] See [1, Sec. 4.2].

Definition 2.31 (Epsilon-invariance over a class)
The formula $\varphi \in \mathrm{Form}(\mathfrak{L}_\varepsilon)$ is *epsilon-invariant over the class* K *of first-order models*, if for every model $\mathfrak{M} \in \mathsf{K}$, for every $a \in {}^\omega M$ and for all choice functions f and g such that $(\mathfrak{M}, f), (\mathfrak{M}, g) \in \mathsf{Ext}(\mathfrak{L}_\varepsilon)$

$$(\mathfrak{M}, f) \models \varphi[a] \quad \text{iff} \quad (\mathfrak{M}, g) \models \varphi[a]$$

holds. (Cf. [16, Def. 1].)

Remark 2.32
The concept above is crucial in [16], where it is shown that epsilon-languages are more expressive than first-order languages over the class of all finite models.

The following proposition is a simple fact about the epsilon-invariance and the intensionally valid sentences.

Proposition 2.33
Let K be a set of first-order models and let $\varphi \in \mathrm{Sent}(\mathfrak{L}_\varepsilon)$. If $\mathsf{K} \underset{\mathrm{Int}}{\models} \varphi$, then φ is epsilon-invariant over the class K.

Proof. Let $\mathfrak{M} \in \mathsf{K}$, $a \in {}^\omega M$ and let f and g be such that $(\mathfrak{M}, f), (\mathfrak{M}, g) \in \mathsf{Ext}(\mathfrak{L}_\varepsilon)$. $(\mathfrak{M}, f) \models \varphi[a]$ and $(\mathfrak{M}, g) \models \varphi[a]$ holds, since f and g, as extensional choice functions, are also intensional choice functions. Hence, φ is epsilon-invariant over K. ∎

3 Proofs

Let us recall sentence (M) from Section 1.3 and let us introduce a notation for its first-order form.

Definition 3.1
Let

$$\mathsf{InvSub}(\varphi, \vartheta)$$

denotes the formula

$$((\exists x)\varphi \wedge (\forall x)(\varphi \to \vartheta)) \vee (\neg(\exists x)\varphi \wedge (\forall x)\vartheta).$$

Our aim is to show, step by step, a meta-equivalence like

$$\vdash \vartheta[(\varepsilon x)\varphi/x] \quad \text{iff} \quad \vdash \mathsf{InvSub}(\varphi, \vartheta)$$

without semantic conditions.

First, note that, in the epsilon-calculus, $\mathsf{InvSub}(\varphi, \vartheta)$ implies $\vartheta[(\varepsilon x)\varphi/x]$ without any assumptions.

Proposition 3.2
If φ and ϑ are monadic predicates of the variable x, then

$$\underset{\mathrm{EC}\varepsilon}{\vdash} \vartheta[(\varepsilon x)\varphi/x] \leftarrow \mathsf{InvSub}(\varphi, \vartheta).$$

Proof. First, let us suppose $(\exists x)\varphi$ and $(\forall x)(\varphi \to \vartheta)$. By the rules of epsilon-calculus, we have

$$\{(\exists x)\varphi\} \underset{\mathrm{EC}\varepsilon}{\vdash} \varphi[(\varepsilon x)\varphi/x] \quad \text{and} \quad \{(\exists x)\varphi, (\forall x)(\varphi \to \vartheta)\} \underset{\mathrm{EC}\varepsilon}{\vdash} \vartheta[(\varepsilon x)\varphi/x].$$

This yields $\underset{\mathrm{EC}\varepsilon}{\vdash} ((\exists x)\varphi \wedge (\forall x)(\varphi \to \vartheta)) \to \vartheta[(\varepsilon x)\varphi/x]$.
Second, suppose $\neg(\exists x)\varphi$ and $(\forall x)\vartheta$. Then

$$\{(\forall x)\vartheta\} \underset{\mathrm{EC}\varepsilon}{\vdash} \vartheta[(\varepsilon x)\varphi/x]$$

implies $\underset{\mathrm{EC}\varepsilon}{\vdash} (\neg(\exists x)\varphi \wedge (\forall x)\vartheta) \to \vartheta[(\varepsilon x)\varphi/x]$. Using the method of proof by cases it follows that the proposition holds. ∎

In order to prove a couple of meta-equivalences between $\vartheta[(\varepsilon x)\varphi/x]$ and $\mathsf{InvSub}(\varphi,\vartheta)$ we need some new syntactic and semantic notions. In the epsilon-language the terms and formulas are defined simultaneously and they are called *well-formed expressions*, or *wf expressions* for short. Every wf expression α has at least one *wf expression construction* $(\alpha_1,\alpha_2,\ldots,\alpha_n)$, where $\alpha_n=\alpha$ and the wf expression α_i is generated by the previous ones by the operators $\neg,\vee,\varepsilon,\mathsf{t}$ following the inductive definition of the terms and formulas (for every $1\le i\le n$).

Definition 3.3
The wf expression α *omits the set* $\mathrm{mat}((\varepsilon x)\varphi)$ if α has a wf expression construction $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ such that

$$\{\alpha_i \mid 1\le i\le n\}\cap \mathrm{mat}((\varepsilon x)\varphi)=\emptyset.$$

Definition 3.4
Let K be a class of first-order models of type t. The formula ϑ is *epsilon-invariant in* $(\varepsilon x)\varphi$ *over* K, if for all models $\mathfrak{M}\in\mathsf{K}$ and for every choice function f,g such that $(\mathfrak{M},f),(\mathfrak{M},g)\in\mathsf{Int}(\mathcal{L}_\varepsilon)$, $f=g$ on the set $(\mathrm{Mat}\setminus\{\mathrm{mat}((\varepsilon x)\varphi)\})\times\mathcal{P}(M)\times{}^{[\omega]}M$ and for every $a\in{}^{\omega}M$

$$(\mathfrak{M},f)\models\vartheta[a] \text{ iff } (\mathfrak{M},g)\models\vartheta[a].$$

The former concept ‘an expression omitting the set $\mathrm{mat}((\varepsilon x)\varphi)$’ is the same as ‘the matrix does not occur in an expression’ which is defined in [13]. Here, we prefer an exact definition that recall the structure of a wf expression by mentioning the construction of the wf expression. The latter concept is a weak version of epsilon-invariance. Clearly, if a formula is epsilon-invariant over a class K, then it is epsilon-invariant in every epsilon-term over the class K. Now we continue with a lemma.

Lemma 3.5
Let the wf expression α omit the set $\mathrm{mat}((\varepsilon x)\varphi)$ and let $(\mathfrak{M},f),(\mathfrak{M},g)\in\mathsf{Int}(\mathcal{L}_\varepsilon)$. If $f(m,S,s)=g(m,S,s)$ for every $m\in\mathrm{Mat}\setminus\{\mathrm{mat}((\varepsilon x)\varphi)\}$, $S\in\mathcal{P}(M)$, and $s\in{}^{[\omega]}M$, then

$$\alpha^{(\mathfrak{M},f)}=\alpha^{(\mathfrak{M},g)}.$$

Proof. The proof goes by structural induction. The crucial case is that of the epsilon-terms. Let $(\varepsilon v_i)\vartheta$ be an epsilon-term.

$$\begin{aligned}((\varepsilon v_i)\vartheta)^{(\mathfrak{M},f)}[a] &= f\big(\mathrm{mat}((\varepsilon v_i)\vartheta),\{u\in M\mid(\mathfrak{M},f)\models\vartheta[a_i^u]\},s\big)\\ &= g\big(\mathrm{mat}((\varepsilon v_i)\vartheta),\{u\in M\mid(\mathfrak{M},f)\models\vartheta[a_i^u]\},s\big) \qquad [*]\\ &= ((\varepsilon v_i)\vartheta)^{(\mathfrak{M},g)}[a].\end{aligned}$$

If the formula ϑ above is epsilon-free, then step * is obviously valid, if ϑ is not epsilon-free, then in step * we can use the induction hypothesis. ∎

Now we prove the formula $\vartheta[(\varepsilon x)\varphi/x] \leftrightarrow \mathsf{InvSub}(\varphi,\vartheta)$ in a given model. The phrase 'ψ is epsilon-invariant in $(\varepsilon x)\varphi$ over the model $\mathfrak{M}$' means ψ is epsilon-invariant in $(\varepsilon x)\varphi$ over the class $\{(\mathfrak{M},f)\in \mathsf{Int}(\mathcal{L}_\varepsilon)\,|\, f\in \mathrm{pr}_2\,\mathsf{Int}(\mathcal{L}_\varepsilon)\}$ with the *fixed* model $\mathfrak{M}$.

Proposition 3.6

Let φ and ϑ be monadic predicates of the variable x. If the formulas ϑ and φ omit the set $\mathrm{mat}((\varepsilon x)\varphi)$, and $\vartheta[(\varepsilon x)\varphi/x]$ is epsilon-invariant in $(\varepsilon x)\varphi$ over the model $\mathfrak{M}$, then for every f such that $(\mathfrak{M},f)\in \mathsf{Int}(\mathcal{L}_\varepsilon)$

$$(\mathfrak{M},f)\models \vartheta[(\varepsilon x)\varphi/x] \rightarrow \mathsf{InvSub}(\varphi,\vartheta)$$

holds.

Proof. Let us suppose that $(\mathfrak{M},f)\models \vartheta[(\varepsilon x)\varphi/x]$. At first, we will prove that

$$(\mathfrak{M},f)\models (\forall x)\neg\varphi \text{ implies } (\mathfrak{M},f)\models (\forall x)\vartheta,$$

then we will prove that

$$(\mathfrak{M},f)\models (\exists x)\varphi \text{ implies } (\mathfrak{M},f)\models (\forall x)(\varphi \longrightarrow \vartheta).$$

(1) Let us suppose that $(\mathfrak{M},f)\models (\forall x)\neg\varphi$. It is needed to prove that $(\mathfrak{M},f)\models (\forall x)\vartheta$, which is equivalent to

$$(\mathfrak{M},f)\models \vartheta[(\varepsilon x)(\neg\vartheta)/x].$$

It is clear that $(\mathfrak{M},f)\models (\forall x)\neg\varphi$ implies $\varphi^{(\mathfrak{M},f)}=\varnothing$, since, for every valuation a

$$\begin{aligned}(\mathfrak{M},f)\models ((\forall x)\neg\varphi)[a] &\text{ iff } (\mathfrak{M},f)\models (\neg\varphi[(\varepsilon x)\varphi/x])[a]\\ &\text{ iff } (\mathfrak{M},f)\models (\neg\varphi)\big[((\varepsilon x)\varphi)^{(\mathfrak{M},f)}[a]\big] \text{ [Sub. Lem.]}\\ &\text{ iff } (\mathfrak{M},f)\not\models \varphi\big[((\varepsilon x)\varphi)^{(\mathfrak{M},f)}[a]\big]\\ &\text{ iff } ((\varepsilon x)\varphi)^{(\mathfrak{M},f)}[a]\notin \varphi^{(\mathfrak{M},f)}[a]\\ &\text{ iff } \varphi^{(\mathfrak{M},f)}=\varnothing.\end{aligned}$$

Let $(\varepsilon x)\psi(x,t_1,\dots,t_n)=(\varepsilon x)\varphi$ where $(\varepsilon x)\psi(x,w_1,\dots,w_n)$ is the matrix of $(\varepsilon x)\varphi$ and $t_1,\dots,t_n$ are its exterior epsilon-terms. Let $b=((\varepsilon x)(\neg\vartheta))^{(\mathfrak{M},f)}$ and let g be such that $(\mathfrak{M},g)\in \mathsf{Int}(\mathcal{L}_\varepsilon)$, and for every $(m,S,s)\in \mathrm{Mat}\times \mathcal{P}(M)\times {}^{[\omega]}M$,

$$g(m,S,s)=\begin{cases} b & \text{if } (m,S,s)=(\mathrm{mat}((\varepsilon x)\varphi),\emptyset,(t_1^{(\mathfrak{M},f)},\dots,t_n^{(\mathfrak{M},f)}))\\ f(m,S,s) & \text{otherwise.}\end{cases}$$

By the previous lemma,

$$\begin{aligned}((\varepsilon x)\varphi)^{(\mathfrak{M},g)} &= g(\mathrm{mat}((\varepsilon x)\varphi),\varphi^{(\mathfrak{M},g)},(t_1^{(\mathfrak{M},g)},\dots,t_n^{(\mathfrak{M},g)}))\\ &= g(\mathrm{mat}((\varepsilon x)\varphi),\varphi^{(\mathfrak{M},f)},(t_1^{(\mathfrak{M},f)},\dots,t_n^{(\mathfrak{M},f)})) \qquad \text{[prev. lem.]}\\ &= g(\mathrm{mat}((\varepsilon x)\varphi),\emptyset,(t_1^{(\mathfrak{M},f)},\dots,t_n^{(\mathfrak{M},f)}))\\ &= b.\end{aligned}$$

Therefore,

$$
\begin{array}{lll r}
(\mathfrak{M},f)\models\vartheta[(\varepsilon x)\varphi/x] & \text{iff} & (\mathfrak{M},g)\models\vartheta[(\varepsilon x)\varphi/x] & \text{[eps.-inv.]}\\
 & \text{iff} & ((\varepsilon x)\varphi)^{(\mathfrak{M},g)}\in\vartheta^{(\mathfrak{M},g)} & \text{[Sub. Lem.]}\\
 & \text{iff} & b\in\vartheta^{(\mathfrak{M},g)} & \text{[def. of } g]\\
 & \text{iff} & b\in\vartheta^{(\mathfrak{M},f)} & \text{[prev. lem.]}\\
 & \text{iff} & ((\varepsilon x)(\neg\vartheta))^{(\mathfrak{M},f)}\in\vartheta^{(\mathfrak{M},f)} & \text{[def. of } b]\\
 & \text{iff} & (\mathfrak{M},f)\models\vartheta[(\varepsilon x)(\neg\vartheta)/x]. & \text{[Sub. Lem.]}
\end{array}
$$

Hence, $(\mathfrak{M},f)\models(\forall x)\vartheta$.

(2) Let us suppose that $(\mathfrak{M},f)\models(\exists x)\varphi$. We will show that

$$(\mathfrak{M},f)\models(\forall x)(\varphi\longrightarrow\vartheta),$$

which is equivalent to

$$(\mathfrak{M},f)\models\varphi[(\varepsilon x)(\neg(\varphi\to\vartheta))/x]\longrightarrow\vartheta[(\varepsilon x)(\neg(\varphi\to\vartheta))/x].$$

For the sake of simplicity, let us denote $(\varepsilon x)(\neg(\varphi\to\vartheta))$ by t. Let us suppose that

$$(\mathfrak{M},f)\models\varphi[t/x].$$

Now, we let $b=t^{(\mathfrak{M},f)}$ and let g be the following function: for every $(m,S,s)\in\mathrm{Mat}\times\mathcal{P}(M)\times{}^{[\omega]}M$,

$$g(m,S,s)=\begin{cases} b & \text{if } (m,S,s)=(\mathrm{mat}((\varepsilon x)\varphi),\varphi^{(\mathfrak{M},f)},(t_1^{(\mathfrak{M},f)},\ldots,t_n^{(\mathfrak{M},f)}))\\ f(m,S,s) & \text{otherwise.}\end{cases}$$

Hence, $((\varepsilon x)\varphi)^{(\mathfrak{M},g)}=b$. By the Substitution Lemma and the assumption, it is clear that $b\in\varphi^{(\mathfrak{M},f)}$. All these imply the following:

$$
\begin{array}{lll r}
(\mathfrak{M},f)\models\vartheta[(\varepsilon x)\varphi/x] & \text{iff} & (\mathfrak{M},g)\models\vartheta[(\varepsilon x)\varphi/x] & \text{[eps.-inv.]}\\
 & \text{iff} & ((\varepsilon x)\varphi)^{(\mathfrak{M},g)}\in\vartheta^{(\mathfrak{M},g)} & \text{[Sub. Lem.]}\\
 & \text{iff} & b\in\vartheta^{(\mathfrak{M},g)} & \text{[def. of } g]\\
 & \text{iff} & b\in\vartheta^{(\mathfrak{M},f)} & \text{[prev. lem.]}\\
 & \text{iff} & t^{(\mathfrak{M},f)}\in\vartheta^{(\mathfrak{M},f)} & \text{[def. of } b]\\
 & \text{iff} & (\mathfrak{M},f)\models\vartheta[t/x]. & \text{[Sub. Lem.]}
\end{array}
$$

Hence, $(\mathfrak{M},f)\models(\varphi\longrightarrow\vartheta)[t/x]$ and finally

$$(\mathfrak{M},f)\models(\forall x)(\varphi\longrightarrow\vartheta)$$

holds.

∎

To do the next step, we omit the epsilon-invariance condition by turning to the meta-level.

Corollary 3.7
Let the monadic predicates φ and ϑ of the variable x omit the set mat$((\varepsilon x)\varphi)$, and let K be a class of first-order models.

(1) If $\vartheta[(\varepsilon x)\varphi/x]$ is epsilon-invariant in $(\varepsilon x)\varphi$ over K, then

$$\mathsf{K} \underset{\mathrm{Int}}{\models} \vartheta[(\varepsilon x)\varphi/x] \leftrightarrow \mathsf{InvSub}(\varphi, \vartheta).$$

(2)
$$\mathsf{K} \underset{\mathrm{Int}}{\models} \vartheta[(\varepsilon x)\varphi/x] \quad \text{iff} \quad \mathsf{K} \underset{\mathrm{Int}}{\models} \mathsf{InvSub}(\varphi, \vartheta).$$

Proof. (1) is implied by Proposition 3.6. (2) By Proposition 2.33, $\mathsf{K} \underset{\mathrm{Int}}{\models} \vartheta[(\varepsilon x)\varphi/x]$ implies that $\vartheta[(\varepsilon x)\varphi/x]$ is epsilon-invariant in $(\varepsilon x)\varphi$ over K. Hence, $\mathsf{K} \underset{\mathrm{Int}}{\models} \mathsf{InvSub}(\varphi, \vartheta)$ follows from (1). The opposite direction holds by Proposition 3.2. ∎

Finally, let us introduce a notation and prove the main theorem. Let Γ be a set of epsilon-invariant sentences. According to the Blass–Gurevich Theorem, for every $\varphi \in \Gamma$ there is a first-order reformulation ψ preserving the validity. Let Γ' be the set of all such reformulations of the members of Γ and let $\mathsf{Mod}^{\mathrm{FO}}(\Gamma)$ be the class of all first-order models of Γ'. Note that, for every choice function f sending elements into M, if $\mathfrak{M} \in \mathsf{Mod}^{\mathrm{FO}}(\Gamma)$, then $(\mathfrak{M}, f) \models \Gamma$.

Theorem 3.8
If the monadic predicates φ and ϑ of the variable x omit the set mat$((\varepsilon x)\varphi)$ and Γ consists of epsilon-invariant sentences, then

$$\Gamma \underset{\mathrm{EC}\varepsilon}{\vdash} \vartheta[(\varepsilon x)\varphi/x] \quad \text{iff} \quad \Gamma \underset{\mathrm{EC}\varepsilon}{\vdash} \mathsf{InvSub}(\varphi, \vartheta).$$

Proof. Let us set $\mathsf{K} = \mathrm{Mod}^{\mathrm{FO}}(\Gamma)$, use Corollary 3.7.2, and apply the completeness property. ∎

If we consider plain first-order sentences for the elements of Γ, then Theorem 3.8 becomes a non-model-theoretic (actually syntactic) proposition on the meaning of epsilon-substitutions.

4 Conclusions

According to the Blass–Gurevich Theorem, if the formula $\vartheta[(\varepsilon x)\varphi/x]$ is epsilon-invariant, then it has a first-order reformulation. In Corollary 3.7.1 it is stated that if $\vartheta[(\varepsilon x)\varphi/x]$ is epsilon-invariant in $(\varepsilon x)\varphi$, and φ and ϑ omit the set mat$((\varepsilon x)\varphi)$, then $\vartheta[(\varepsilon x)\varphi/x]$ has an *explicit* first-order reformulation. Such a reformulation in the case of Russell's iota was considered the contextual meaning, thus in addition to its referential meaning, given by the interpretation, $(\varepsilon x)\varphi$ has also a contextual meaning and it is explicitly stated in the meta-equivalence

$$\underset{\mathrm{EC}\varepsilon}{\vdash} \vartheta[(\varepsilon x)\varphi/x] \quad \text{iff} \quad \underset{\mathrm{EC}\varepsilon}{\vdash} ((\exists x)\varphi \wedge (\forall x)(\varphi \rightarrow \vartheta)) \vee (\neg(\exists x)\varphi \wedge (\forall x)\vartheta).$$

REMARK 4.1
Note that, the reformulation above can easily be explained at the meta-level. Let the formula φ be the formal translation of the meta-language predicate F. The critical axiom,

$$(\exists x)\varphi \to \varphi[(\varepsilon x)\varphi/x],$$

gives the usual meaning of the term $(\varepsilon x)\varphi$: 'an individual object which is F, provided that there is at least one F in the universe of discourse'. Hence, it is plausible to formulate the natural-language counterpart of Theorem 3.8 in the following form.

PROPOSAL 4.2
'An object which is F—if there is any F—is G' means 'there is an F and if something is an F, then it is also a G, or there is no F, but everything is G', provided that the former has some contextual meaning.

REMARK 4.3
Kneebone in [10] took two remarkable notes on Hilbert's epsilon-operator. He considered the theory of epsilons as the generalization of Russell's Theory of Descriptions and he interprets epsilon-terms as indefinite descriptions:

> An ε-term may be thought of as formalizing an indefinite description, somewhat as an ι-term formalizes a definite description[…]. ([10, p. 101, ftn. 1])

Moreover, he pointed out that

> [a critical epsilon axiom] is a weakened version of the ι-schema (p. 94), with the second uniqueness formula left out of account. ([10, p. 101])

If one accepts Kneebone's interpretation, then Proposal 4.2 gives a possible contextual meaning of indefinite descriptions.

On this basis, we are ready to examine the cases of attributive and non-attributive descriptions. However, these investigations belong rather to the philosophy of language and should take place within intensional or philosophical logic. With our results in Proposition 3.6 and Theorem 3.8, we stay within the realm of mathematical logic taken in the narrow sense.

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